Inverse Problems Symposium (IPS) 2013
Huntsville, Alabama
9-11 June 2013

Extended Abstract Entitled

"A FUNDAMENTAL STEP TOWARD A NONLINEAR CALIBRATION METHOD FOR INVERSE HEAT CONDUCTION"

Authors: Dr. J. I. Frankel, Professor, Associate Fellow, AIAA
Dr. M. Keyhani, Professor, Fellow, ASME
Mechanical, Aerospace and Biomedical Engineering Department
University of Tennessee, Knoxville,
Knoxville, Tennessee (USA) 37996-2210
Emails: jfranke1@utk.edu, keyhani@utk.edu
Tel: 1.865.974.5115, Fax:1.865.974.5274

EXTENDED ABSTRACT: This talk describes a surface heat flux calibration method applicable to high-speed, aero-thermodynamics based on in-depth temperature measurements. In hypersonic flight, high surface temperatures and heat fluxes ensue on both the external portion of the flight vehicle and within the combustor. These locations require specialized materials for thermal protection in the hot structure. Leading edges and nose tips are often composed of composite materials such as carbon-carbon, C-C. On the other hand, combustors may use thermal barrier coatings (TBC's) such as zirconium oxide, ZrO_2. In both situations, the objective involves protecting and maintaining the integrity of the underlying structure. Various material solutions have been proposed depending on the flight speed, flight scenerio, and mission objectives. Irrespective of the internal or external flow nature, the accurate estimation of the surface heat flux (W/m^2) and/or the total heat transfer (W) are important for the purpose of material evaluation. Both ground and flight tests should be kept in mind when designing the predictive tools. Ground-based testing can involve either short-time or long-time heat flux exposures to the sample. In short-time experiments, semi-infinite analysis is often used for estimating surface heat flux as in a thin-film resistive temperature gauge and/or co-axial thermocouple [1-5]. In other cases, such as involving arcjets, full thermal penetration to the back surface occurs and thus must be accounted [6-9] in the inverse analysis.

A new calibration methodology is presently under development at the University of Tennessee, Knoxville for estimating surface heat fluxes based on in-depth temperature measurements [5,8,9] applicable to coupon or plug geometries composed of single or multi-regions and/or orthotropic materials. References 5,8,9 describe the calibration concept in the context of linear analysis for both a surface mounted probe [5]; and, an in-depth placed probe [8,9]. Reference 9 presents experimental verification of the analysis described in Ref. 8 in the context of low temperatures.

The next step toward developing a comprehesive, and unified treatment involves extending the approach to fully nonlinear problems involving a large temperature variation. Under this condition, constant thermophysical properties would be an unreasonable assumption to impose. Linear functional equations provide easy access to the frequency domain through either a Laplace or Fourier transform. In the frequency domain, geometric, thermophysical and sensor properties can be analytically eliminated when establishing an input-output view to the inverse process. Lacking this fundamental transform tool for nonlinear problems requires clever and creative approaches that can combine physics and applied mathematics. The novel framework described in Refs. [5,8,9] require a linear mathematical model. This

presentation proposes a new fully nonlinear calibration approach to inverse heat conduction based on a series of observations focused on reformulating the heat equation in terms of various combinations of primitive; and, combined thermophysical properties. That is, a new mathematical formulation is proposed predicated on a series of observations associated with expressing the heat euqation in terms of thermophysical properties. Through these properties, a pattern emerges suggesting a new calibration integral equation for inverse heat conduction in the presence of temperature varying properties. This extended concept combines the power of linear analysis with a series of physical observations while retaining the elegance of the input-output format previously described [5,8,9].

In the context of linear theory, sensor characterization, sensor positioning and thermophysical properties (assumed constant) are implicitly contained in the calibration integral equation that relates the unknown surface (net) heat flux to the in-depth temperature measurements; and, the calibration surface heat flux. Hence, the calibration equation is expressed only in terms of input-output variables (measured temperatures and net surface heat fluxes) and applicable to both in-depth (sensor placement) and surface (sensor placement) analyses. The resulting Volterra integral equation of the first kind contains only discrete data. Being ill-posed, regularization is required. The present approach is based on a local-future information method [5,8,9,10]. The optimal regularization parameter is estimated based on interrogating the global residuals and their randomness (not conventional norms). The conflict between bias and variance can be exploited for estimating the optimal regularization parameter. Consider a semi-infinite, one-dimensional (x≥0) region. Thus, the heat equation for this geometry is given by

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}(x,t)=\frac{\partial^2 T}{\partial x^2}(x,t), \quad (x,t)\geq 0, \tag{1a}$$

subject to the boundary and initial conditions given as

$$q_s''(t)=q''(0,t)=-k\frac{\partial T}{\partial x}(0,t),$$

$$\underset{x\to\infty}{Lim}\, T(x,t)=T_o,\ \ t>0, \tag{1b-d}$$

$$T(x,0)=T_o, \quad x\geq 0,$$

while the resulting calibration formulation has been shown to be expressible as [8,9]

$$\int_{u=0}^{t} q_r''(0,u)T_{tc,c}(b,t-u)du=\int_{u=0}^{t} q_c''(0,u)T_{tc,r}(b,t-u)du, \quad t\geq 0 \tag{1e}$$

where $T_{tc,c}$ is the calibration (measured) temperature at some depth denoted as b≥0, q_c'' is the net surface heat flux from the calibration run; $T_{tc,r}$ is the measure in-depth temperature of the run associated with the desired predictive heat flux; and, q_r'' is the desired net surface heat flux to be predicted. Equation (1e) assumes T_o = 0°C or alternatively we can be interpret Eq. (1e) in terms of a reduced temperature ($T-T_o$). The identical calibration integral equation results even if the spatial region is a slab of width L with an adiabatic back condition or one in which the heat transfer coefficient does not change between runs. Equation (1e) does not require the apriori specification of thermophysical properties, probe positioning, nor sensor response time parameters. The key input issue lies in the accurate estimation of the net surface heat flux during the calibration stage.

For the present talk, consider the nonlinear heat equation in a semi-infinite domain as

$$\rho c(T)\frac{\partial T}{\partial t}(x,t)=\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}(x,t)\right), \quad (x,t)\geq 0, \tag{2a}$$

subject to the boundary and initial conditions given by

$$q_s''(t) = q''(0,t) = -k(T(0,t))\frac{\partial T}{\partial x}(0,t),$$

$$\lim_{x\to\infty} T(x,t) = T_o, \; t > 0, \tag{2b-d}$$

$$T(x,0) = T_o, \quad x \ge 0,$$

respectively. An important aspect of the present study lies in the variation of thermophysical properties as a function of temperature. Figures 1a-d display various normalized thermophysical properties using the property average defined as

$$\bar{\lambda} = \frac{1}{T_{\max} - T_{\min}} \int_{z=T_{\min}}^{T_{\max}} \lambda(z)dz \tag{3}$$

where the primitive properties (k,c,α,β) are expressed in terms of an expansion in the form

$$\lambda(T) = \lambda_o + \lambda_1(T - T_{ref}) + \lambda_2(T - T_{ref})^2 + \ldots \tag{4}$$

Here, λ represents an arbitrary property; and, T_{min}=20°C and T_{max}=800°C. Figures 2-5 display alternative property combinations that are mathematically obtained and assembled through various property transforms to demonstrate the proposed concept. These transforms and underlying assumptions will allow for a property linearization of the heat equation. This observation is the first step toward developing a generalized calibration approach for inverse heat conduction.

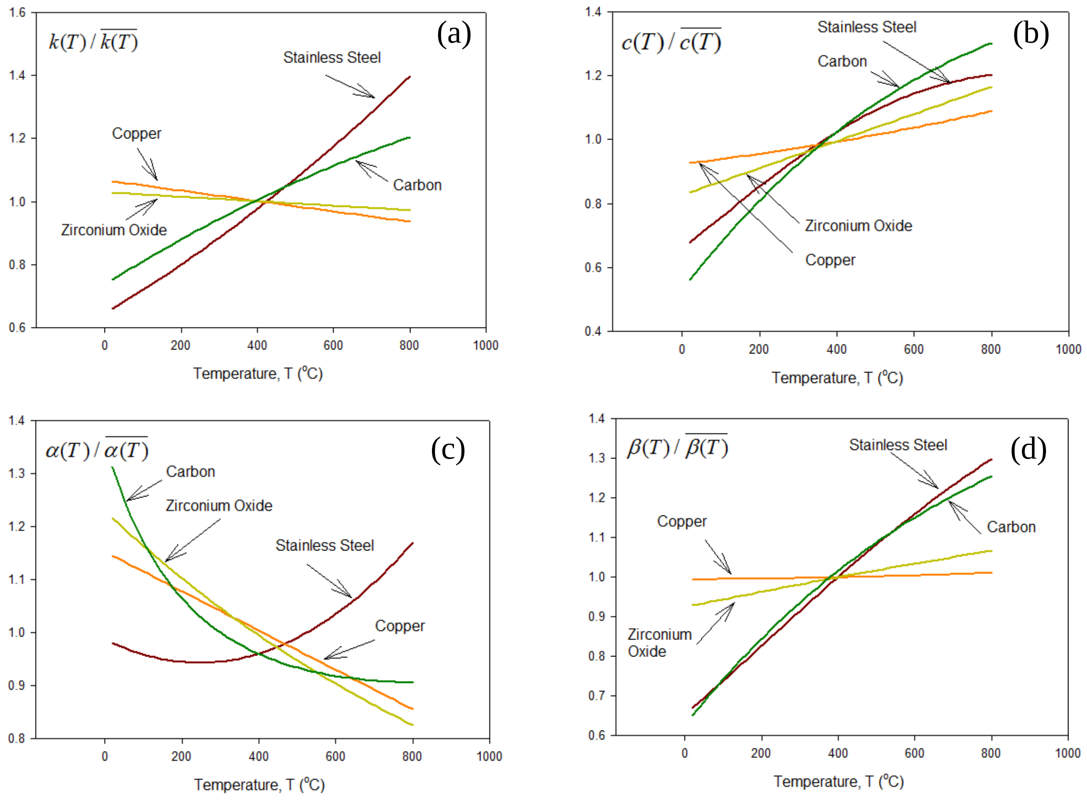

Figure 1: Various thermophysical property variations for (a) thermal conductivity, (b) specific heat, (c) thermal diffusivity, and (d) thermal effusivity as a function of temperature for four materials.

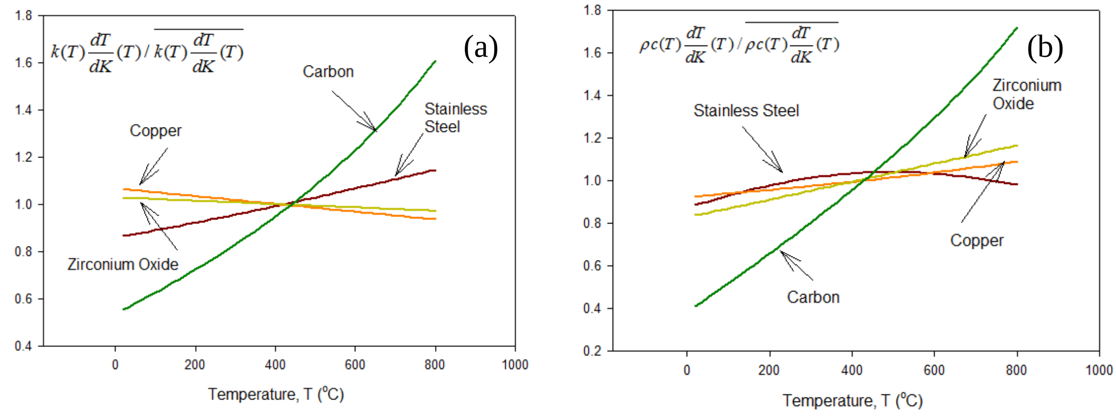

Figure 2: Thermal conductivity transform.

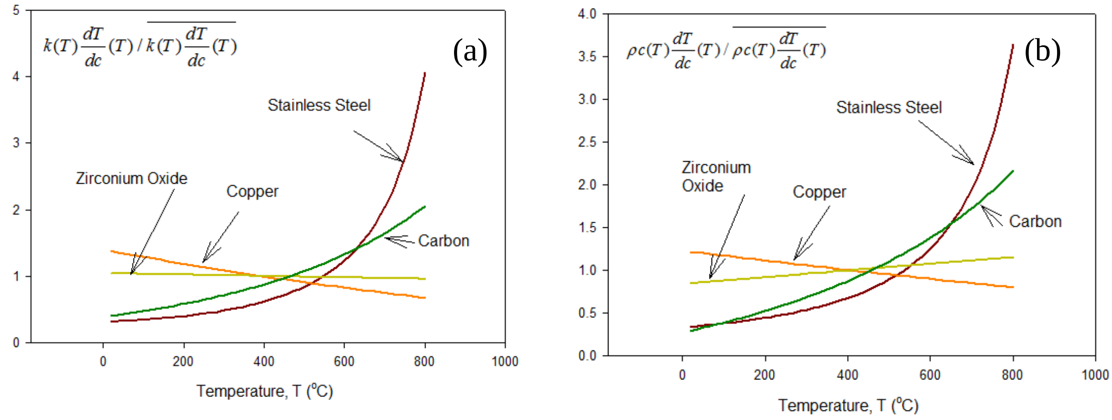

Figure 3: Capacitance transform.

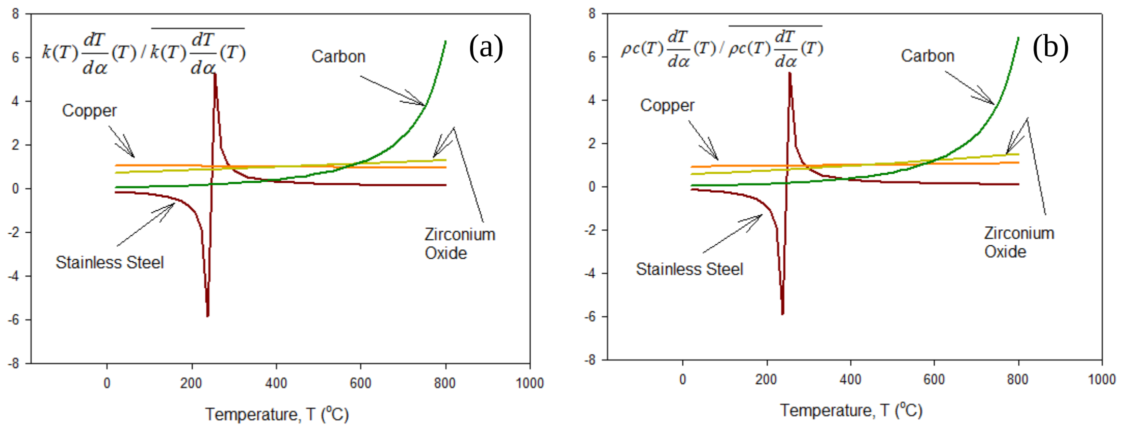

Figure 4: Thermal diffusivity transform.

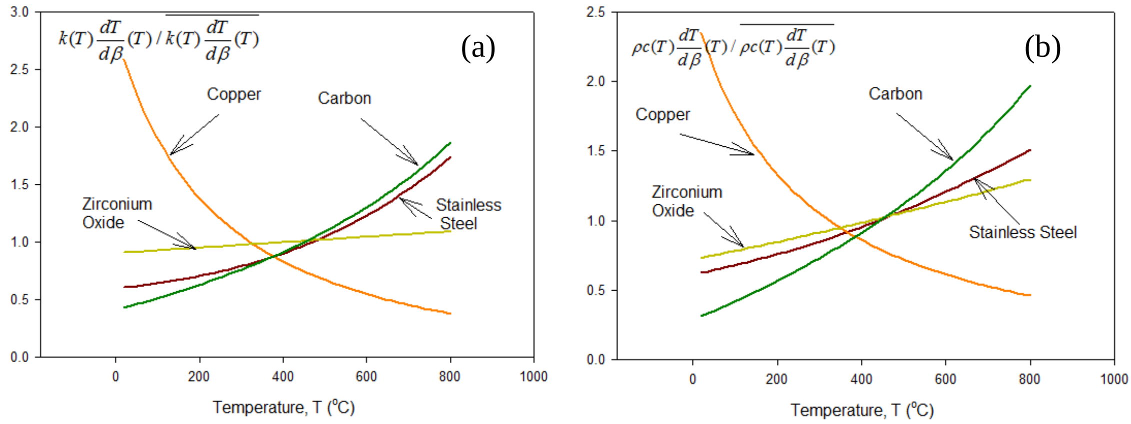

Figure 5: Thermal effusivity transform.

With these new property definitions, we can develop a series of linearized heat equations for investigation. The linearization is especially helpful when the observable variation about the average is small, say ±10% (see Figs 1-5). For example, Figs. 2a,b show that both copper and ZrO_2 display such a feature in the context of the "conductivity transform". As shown in these figures such cases occur depending on the property definition. Additionally, we can call upon the Kirchhoff transform as a tool for linearization if the thermal diffusivity is relatively constant (see Fig. 1c and stainless steel for T<600°C). It should be noted that numerous transforms can be proposed as shortly demonstrated. For example, consider the thermal *Conductivity Transform* as defined in the following manner by chain rule calculus:

Conductivity Transform :

$$\frac{\partial T}{\partial x}(x,t)=\frac{dT}{dk}\frac{\partial k}{\partial x},\quad \frac{\partial T}{\partial t}(x,t)=\frac{dT}{dk}\frac{\partial k}{\partial t},\qquad k(T)=k(T(x,t))=K(x,t). \tag{4a,b}$$

The heat equation can alternatively be expressed as

$$\rho c(T)\frac{dT}{dk}\frac{\partial K}{\partial t}(x,t)=\frac{\partial}{\partial x}\left(k(T)\frac{dT}{dk}\frac{\partial K}{\partial x}(x,t)\right),\ (x,t)\geq 0. \tag{4c}$$

For the moment, consider the situation when

$$\beta_k=\rho c(T)\frac{dT}{dk}\approx constant,\quad \gamma_k=k(T)\frac{dT}{dk}\approx constant \tag{4d,e}$$

then

$$\frac{1}{\varepsilon_k}\frac{\partial K}{\partial t}(x,t)=\frac{\partial^2 K}{\partial x^2}(x,t),\ (x,t)\geq 0 \tag{4f}$$

where $\varepsilon_k=\frac{\gamma_k}{\beta_k}$. The boundary conditions in the transformed variable become

$$q_s''(t)=q''(0,t)=-\gamma_k\frac{\partial K}{\partial x}(0,t),\qquad \lim_{x\to\infty}K(x,t)=k(T_o),\ t>0, \tag{4g,h}$$

with the transformed initial condition

$$K(x,0)=k(T_o),\quad x\geq 0. \tag{4i}$$

This formulation indicates the procedure for proposing the combined properties displayed Figures 2-5. Observe that Eq. (4f) is similar to Eq. (1a); and, hence we can derive a similar calibration equation as given in Eq. (1e). Table 1 presents a summary of calibration equations based on Kirchhoff and the various thermophysical property transforms associated with Figs. 1-4.

Table 1: Summary of calibration equations based on properties.

$$\textit{Linear Heat Equation } (T \textit{ data}):$$

$$\int_{u=0}^{t} q_r''(0,u)\left(T_{tc,c}(b,t-u)-T_{oc}\right)du=\int_{u=0}^{t} q_c''(0,u)\left(T_{tc,r}(b,t-u)-T_{or}\right)du,\quad t\ge 0$$

$$\textit{Kirchhoff Transform} / \alpha(T)\approx\alpha \ (T \textit{ data}):$$

$$\int_{u=0}^{t} q_r''(0,u)\Psi_c(b,t-u)du=\int_{u=0}^{t} q_c''(0,u)\Psi_r(b,t-u)du,\quad t\ge 0$$

$$\textit{Conductivity Transform } (T \textit{ data}):$$

$$\int_{u=0}^{t} q_r''(0,u)\left(k_c(b,t-u)-k_{oc}\right)du=\int_{u=0}^{t} q_c''(0,u)\left(k_r(b,t-u)-k_{or}\right)du,\quad t\ge 0$$

$$\textit{Capacitance Transform } (T \textit{ data}):$$

$$\int_{u=0}^{t} q_r''(0,u)\left(c_c(b,t-u)-c_{oc}\right)du=\int_{u=0}^{t} q_c''(0,u)\left(c_r(b,t-u)-c_{or}\right)du,\quad t\ge 0$$

$$\textit{Diffusivity Transform } (T \textit{ data}):$$

$$\int_{u=0}^{t} q_r''(0,u)\left(\alpha_c(b,t-u)-\alpha_{oc}\right)du=\int_{u=0}^{t} q_c''(0,u)\left(\alpha_r(b,t-u)-\alpha_{or}\right)du,\quad t\ge 0$$

Each property in the defined transform provided in Table 1 can be replaced by a series representation as given in Eq. (4) possessing different but known coefficients. Observe the common functional form produced under the assumption as explained in Eq. (4d,e) for the conductivity transform. An infinite number of choices exist. With Table 1 results as a physical guide, an alternative and generalized calibration integral equation to be proposed is now expressed as

$$\int_{u=0}^{t} q_r''(0,u)\left(a_0+T_c(b,t-u)+a_2T_c^2(b,t-u)+\ldots-T_{oc}\right)du=$$
$$\int_{u=0}^{t} q_c''(0,u)\left(a_0+T_r(b,t-u)+a_2T_r^2(b,t-u)+\ldots-T_{or}\right)du,\quad t\ge 0 \tag{5}$$

where the unknown coefficients $a_0, a_2, a_3, \ldots$ are to be determined through additional calibration runs.

This talk will present and focus on the following:

- Concept development for one-dimensional one-probe geometries (Eq. (1e));
- Elucidation of the implemented local-future time method and using residual randomness as the measure for obtaining the optimal regularization parameter;
- Presentation of a one-dimensional generalization to fully nonlinear systems applicable to plugs;
- Presentation of both theoretical and experimental results for various cases; and,
- Presentation of the developing and expanding University of Tennessee, Knoxville experimental facilities (electrical sandwich heating facility and laser testing facility (500W, 0.91μm)).

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